HW VII

1 (a) In terms of sequences, state the density result for $\mathbb{Q}$. Do the same for $\mathbb{R} \setminus \mathbb{Q}$.

(b) State the sequential criterion for $\lim_{x \to x_0} f(x) = \begin{cases} l \\ +\infty \\ -\infty \end{cases}$.

(c) State Cauchy criterion for sequence

These results may be helpful for Q2, Q3, Q4 below.

2* Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x) = \begin{cases} 6x - 8 & \forall x \in \mathbb{Q} \\ \frac{7}{x} - 7 & \forall x \notin \mathbb{Q} \end{cases}$$

Find all x_0 for which $\lim_{x \to x_0} g(x)$ exists in $\mathbb{R}$

3* Let $h : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$, $l \in \mathbb{R}$.

Show that $\lim_{x \to x_0} f(x) = l$ iff

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - f(u)| < \varepsilon$ whenever $x, u \in V_\delta(x_0) \setminus \{x_0\}$.

4* Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathbb{Q}$-linear in the sense that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ $\forall \alpha, \beta \in \mathbb{Q}$ and $\forall x, y \in \mathbb{R}$. Suppose $\lim_{x \to 0} f(x) = L \in \mathbb{R}$. Show that f is continuous at any $x_0 \in \mathbb{R}$ in the sense that $\lim_{x \to x_0} f(x) = f(x_0)$.

5. Let $I=(a,b) \subseteq (0,\infty)$ be an interval of length 1 (or finite length). Let $n \in \mathbb{N}$ & $Z_n = \{m \in \mathbb{N} : \frac{m}{n} \in I\}$, $Y_n = \{x \in I \cap \mathbb{Q} : x = \frac{m}{n}$ with some $m \in Z_n\}$

and
$B_n := \{x \in I \cap \mathbb{Q} : x \triangleq \frac{m}{n}$ (canonical representation, i.e. m, n have non-trivial common divisor), with some $m \in Z_n\}$

Show that

1) Z_n is bounded and is a finite set;

2) Y_n and B_n are finite sets;

3) $I \cap \mathbb{Q} = \bigcup_{n \in \mathbb{N}} B_n$;

4). Let $x_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$ (positive irrational), $I := (x_0 - \frac{1}{2}, x_0 + \frac{1}{2}) \cap \mathbb{R}^+$ and let $N \in \mathbb{N}$,

$\delta = \min\{\frac{1}{2}, \text{dist}(x_0, \bigcup_{n=1}^{N} B_n)\}$, where B_n is defined as before. Then (why?)

$\delta > 0$ and if $0 < x \in V_\delta(x_0) \setminus \mathbb{Q}$ and $x \triangleq \frac{m}{n}$ (in canonical representation) then $n > N$.

6). The Thomae function $f : (0,\infty) \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} \frac{1}{n} & x \triangleq \frac{m}{n} \text{ (canonical rep.)}, x \in \mathbb{Q} \cap \mathbb{R}^+ \\ 0 & x \in \mathbb{R}^+ \setminus \mathbb{Q} \end{cases}$$

is cts at any $x_0 \in \mathbb{R}^+ \setminus \mathbb{Q}$.